Exactly solvable models of quantum mechanics including fluctuations in the framework of representation of the wave function by random process

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Abstract

The problem of quantum harmonic oscillator with "regular+random" square frequency, subjected to "regular+random external force, is considered in framework of representation of the wave function by complex-valued random process. Average transition probabilities are calculated. Stochastic density matrix method is developed, which is used for investigation of thermodinamical characteristics of the system, such as entropy and average energy.

Introduction

Recently there have been published a great amount of papers [1] concerned with the "quantum chaos", i.e. with the quantum analogues of classical systems possessing the dynamic chaos features. The investigations are conducted along the different directions, such as analysis of the energy levels distribution; definition and calculation of quantities which are responsible in the quantum systems for the presence of chaos (corresponding to the classical Lyapunov exponents and KS-entropy); study of localization and delocalization of wave functions around classical orbits; etc. It is worth saying that though in most of the cases mentioned above one is faced with

the necessity to describe a quantum system statistically, so far there was not paid much attention to a stochastic behavior of the wave function itself.

Many problems of great importance in the field of the nonrelativistic quantum mechanics, such as description of Lamb shift, spontaneous transitions in atoms, etc., remain unsolved due to the fact that the concept of physical vacuum has not been considered within the framework of the standard quantum mechanics. It is obvious that a quantum object immersed into the physical vacuum is an open system. There exist various approaches [2] to the description of such systems, mainly in application to the problem of continuous measurements. One of them is based on the consideration of the wave function as a random process, for which a stochastic differential equation (SDE) is derived. But the equation is obtained by the method which is extremely difficult for application even in case of comparatively simple type of interaction between the system and the environment, so that some new ideas are needed [3]-[5]. Moreover sometimes it becomes necessary to consider the wave function as a random process even in closed systems (for example, when a classical analogue of the quantum system has the features of the dynamical chaos) [6]-[8].

To be able to describe the cases mentioned above, in the present paper we propose a radically new scheme of derivation of the evolution equation for a nonrelativistic quantum system, interacting with the thermostat (in particular with the physical vacuum), with the wave function represented by a complex-valued random process. The main idea of the new representation may be described as follows: a potential energy of the system "quantum object + thermostat" is assumed to be a random function. This is the case where the Schrödinger equation may be used only locally on small time intervals and may provide good phenomenological models for some problems, which are solvable within the framework of the multiparticle Schrödinger equation.

In the majority of cases the main tool for investigating the particular problems is the perturbation theory, which sometimes fails to provide an adequate description of a real physical phenomenon. In the present paper the influence of the thermostat on an elementary process is considered nonperturbatively within the framework of randomly wandering one-dimensional quantum harmonic oscillator model. Average transition probabilities for the parametric oscillator are calculated, exact representations are found for both widening and shift (analogous to the Lamb shift) of the energy level of the oscillator submerged into the thermostat (vacuum) as well as the entropy of an individual quantum state is calculated.

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1   Description of the problem

We shall consider the closed system "quantum object + thermostat" within the framework of a complex-valued probability process representation. The realization of the process is a wave functional $\Psi_{stc}(x, t|\{\vec{\zeta}\})$, defined on $L_2\left(R^1 \otimes R_{\{\vec{\zeta}\}}\right)$ space (extended space), where $\vec{\zeta}(t)$ denotes a many-dimensional complex-valued random process. Time evolution of the wave functional is governed by the equation

$$i\partial_t \Psi_{stc} = \hat{H}\Psi_{stc}, \tag{1.1}$$

where one-dimensional Hamilton function $\hat{H}$ is assumed to be quadratic over the space variable

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\Omega^2(t)x^2 - F(t)x, \tag{1.2}$$

and the functions $\Omega^2(t)$ and $F(t)$ are random functions of time. Let them have the form

$$\Omega^2(t) = \Omega_0^2(t) + \sqrt{2\epsilon_1 p_1}f_1(t)\Theta(t - t_1),$$
$$F(t) = F_0(t) + \sqrt{2\epsilon_2 p_2}f_2(t)\Theta(t - t_2), \tag{1.3}$$

where $\Omega_0^2(t)$ and $F_0(t)$ are regular (nonrandom) functions and $f_1(t), f_2(t)$ are independent Gaussian random processes with the zero mean and $\delta-$shaped correlators

$$< f_i(t)f_j(t') > = \delta_{ij}\delta(t - t'), \quad i, j = 1, 2. \tag{1.4}$$

Constants ϵ_i, $i = 1, 2$ control the power of forces $f_i(t)$, $t = 1, 2$. Functions $p_i(t)$, $i = 1, 2$ are assumed nonnegative: $p_1, p_2 \geq 0$. Step-function $\Theta(x)$ is defined by

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases} \tag{1.5}$$

Let us assume that the following asymptotic conditions hold

$$\Omega_0(t) \xrightarrow[t \to \pm\infty]{} \Omega_{\left(\begin{smallmatrix} in \\ out \end{smallmatrix}\right)}, \quad F_0(t) \xrightarrow[t \to \pm\infty]{} 0, \quad p_i(t) \xrightarrow[t \to \pm\infty]{} 0, \ i = 1, 2. \tag{1.6}$$

which guarantee that the autonomous states $\phi_n^{in}(x, t)$ exist as $t \to -\infty$

$$\phi_n^{in}(x, t) = e^{-i(n+1/2)\Omega_{in}t}\phi_n^{in}(x),$$
$$\phi_n^{in}(x) = \left(\frac{1}{2^n n!}\sqrt{\frac{\Omega_{in}}{\pi}}\right)^{1/2} e^{-\Omega_{in}x^2/2}H_n(\sqrt{\Omega_{in}}x), \tag{1.7}$$

where $\phi_n^{in}(x)$ is the wave function of a stationary oscillator and $H_n(x)$ is the Hermitian polynomial. It also follows from (1.6) that autonomous states $\phi_n^{out}(x,t)$, which are obtained from (1.7) by a substitution of Ω_{in} by Ω_{out}, exist in the limit $t \to +\infty$ as well. Θ-functions in (1.3) reflect the fact that the random processes $f_1(t)$ and $f_2(t)$ are activated at the moments t_1 and t_2, respectively. If necessary, the functions p_1 and p_2 may be chosen having the form which prevents the jumps of Ω and F when the noise is activating. The moments t_1 and t_2 are assumed to be finite to make the following inference correct. The aim of the paper is to find the average probabilities W_{nm} of transitions from the initial stationary states $\phi_n^{in}(x,t)$ to the final ones $\phi_m^{out}(x,t)$ when the evolution is governed by Hamilton function (1.2). Exact mathematical definition of W_{nm} will be given in the next section.

2 Formal expressions for the wave functional and transition probabilities

Proposition 2.1. *The formal solution of the problem (1.1)-(1.2) may be written down explicitly for arbitrary $\Omega^2(t)$ and $F(t)$. It has the following form*

$$\Psi_{stc}(x,t|\{\vec{\zeta}\}) = \frac{1}{\sqrt{r}} exp\left\{ i\left[\dot{\eta}(x-\eta) + \frac{\dot{r}}{2r}(x-\eta)^2 + \sigma \right] \right\} \chi\left(\frac{x-\eta}{r}, \tau \right), \qquad (2.8)$$

where function $\chi(y,\tau)$ satisfies the Schrödinger equation for a harmonic oscillator with the constant frequency Ω_{in}

$$i\frac{\partial \chi}{\partial \tau} = -\frac{1}{2}\frac{\partial^2 \chi}{\partial^2 y} + \frac{\Omega_{in}^2 y^2}{2}\chi,$$

function $\eta(t)$ is a solution of the classical equation of motion for the oscillator with the frequency $\Omega(t)$, subjected to the external force $F(t)$:

$$\ddot{\eta} + \Omega^2(t)\eta = F(t), \quad \eta(-\infty) = \dot{\eta}(-\infty) = 0, \qquad (2.9)$$

$\sigma(t)$ is a classical action, corresponding to the solution $\eta(t)$

$$\sigma(t) = \int_{-\infty}^{t} \left[\frac{1}{2}\dot{\eta}^2 - \frac{1}{2}\Omega^2\eta^2 + F\eta \right] dt', \qquad (2.10)$$

and $r(t)$ and $\tau(t)$ are expressed in terms of the solution $\xi(t)$ of the homogeneous equation, corresponding to (2.9)

$$\ddot{\xi} + \Omega^2(t)\xi = 0, \quad \xi(t) \underset{t \to -\infty}{\sim} e^{i\Omega_{in}t} \qquad (2.11)$$

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as

$$\xi(t) = r(t)e^{\gamma(t)}, \quad r(t) = |\xi(t)|, \quad \tau = \gamma(t)/\Omega_{in}.$$

There was introduced a special designation $\vec{\zeta} = (\xi, \eta)$ for the set of functions $\xi(t)$ and $\eta(t)$ from (2.8).

Proof. The proof is based on the substitutions first used in [9] and may be performed by the explicit verification under the only suggestion that all the executed manipulations are legal. $\triangle$

The set of solutions of type (2.8) which is important for the following considerations in this paper is obtained from (2.8) after the substitution of $\chi(y, \tau)$ by $\phi_n^{in}(y, \tau)$ from (1.7). It is thus defined as

$$\Psi_{stc}^{(n)}(x, t|\{\vec{\zeta}\}) = \frac{1}{\sqrt{r}}exp\left\{i\left[\dot{\eta}(x - \eta) + \frac{\dot{r}}{2r}(x - \eta)^2 + \sigma\right]\right\}\phi_n^{in}(\frac{x - \eta}{r}, \tau), \qquad (2.12)$$

$$n = 1, 2, ...$$

The main properties of the set of functionals (2.12), which are important in what follows are

1. For any n the functional $\Psi_{stc}^{(n)}(x, t|\{\vec{\zeta}\})$ reduces to the autonomous state $\phi_n^{in}(x, t)$ in the limit $t \to -\infty$.

2. For any fixed $\vec{\zeta}$ elements of the set (2.12) are mutually orthogonal in the sense of $L_2(R^1)$, space of square-integrable functions:

$$\int_{-\infty}^{\infty} \Psi_{stc}^{(n)}(x, t|\{\vec{\zeta}\})\overline{\Psi_{stc}^{(m)}(x, t|\{\vec{\zeta}\})} \, dx = \delta_{nm}, \qquad (2.13)$$

where a bar denotes the complex conjugation procedure and $\delta_{nm} = 1$, for $n = m$ and $\delta_{nm} = 0$ for $n \neq m$.

Definition 2.1. *Average probabilies W_{nm} of transitions from the states $\Psi_{stc}^{(n)}(x, t|\{\vec{\zeta}\})$ to the stationary ones $\phi_m^{out}(x, t)$ in the limit $t \to +\infty$ are defined by*

$$\Psi_{stc}^{(n)}(x, t|\{\vec{\zeta}\}) = \sum_{m=0}^{\infty} c_{nm}(t|\{\vec{\zeta}\})\phi_m^{out}(x, t), \qquad (2.14)$$

$$W_{nm} = \lim_{t \to +\infty} \left\langle |c_{nm}|^2 \right\rangle, \qquad (2.15)$$

where the symbol $< .. >$ denotes the procedure of averaging with respect to f_1 and f_2.

Definition 2.2. *The generating function $I_{stc}(z_1, z_2, t|\{\vec{\zeta}\})$ for coefficients c_{nm} is defined by the expression*

$$I_{stc}(z_1, z_2, t|\{\vec{\zeta}\}) = \int_{-\infty}^{+\infty} dx \overline{\Psi_{out}(\bar{z}_1, x, t)} \Psi_{stc}(z_2, x, t|\{\vec{\zeta}\}), \qquad (2.16)$$

where

$$\Psi_{stc}(z, x, t|\{\vec{\zeta}\}) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \Psi_{stc}^{(n)}(x, t|\{\vec{\zeta}\}),$$

$$\Psi_{out}(z, x, t) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n^{(out)}(x, t),$$

so that

$$c_{nm}(t|f_1, f_2) = \left. \frac{\partial^{n+m} I_{stc}}{\partial z_1^n \partial z_2^m} \right|_{z_1=z_2=0}. \qquad (2.17)$$

Proposition 2.2. *Explicit expression for the generating function (2.16) is given by*

$$I_{stc}(z_1, z_2, t|\{\vec{\zeta}\}) = \left(\frac{2\sqrt{\Omega_{in}\Omega_{out}}}{K\xi} \right)^{1/2} exp\{Az_1^2 + Bz_2^2 + Cz_1z_2 + Dz_1 + Lz_2 + M\} \quad (2.18)$$

where the following notations are made

$$A = \frac{1}{2} e^{2i\Omega_{out}t} \left(\frac{2\Omega_{out}}{K} - 1 \right), \quad B = \frac{1}{2} \left(\frac{2\Omega_{in}}{K\xi^2} - e^{-2i\gamma} \right),$$

$$C = \frac{2\sqrt{\Omega_{in}\Omega_{out}}}{K\xi} e^{i\Omega_{out}t}, \quad L = -\frac{\sqrt{2\Omega_{in}}}{K\xi} (\Omega_{out}\eta - i\dot{\eta}),$$

$$D = \sqrt{2\Omega_{out}} e^{i\Omega_{out}t} \left[\left(1 - \frac{\Omega_{out}}{K} \right) \eta + \frac{i\dot{\eta}}{K} \right]$$

$$M = \frac{\Omega_{out}}{2} \left(\frac{\Omega_{out}}{K} - 1 \right) \eta^2 - \frac{1}{2K} \dot{\eta}^2 - \frac{i\Omega_{out}}{K} \eta\dot{\eta} + i \left(\frac{1}{2}\Omega_{out}t + \sigma \right),$$

$$K = -i\frac{\dot{\xi}}{\xi} + \Omega_{out}.$$

Proof. The proof is carried out by the direct summation of the series over Hermitian polynomials followed by the calculation of the Gaussian integral. $\triangle$

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Expanding the expression (2.18) in z_1 and z_2 powers, we obtain the coefficients c_{nm}. The first several of them are given below

$$c_{00} = \left(\frac{2\sqrt{\Omega_{in}\Omega_{out}}}{K\xi}\right)^{1/2} e^M, \quad c_{01} = Dc_{00}, \quad c_{10} = Lc_{00}, \tag{2.19}$$

$$c_{11} = (C + DL)c_{00}, \quad c_{20} = \sqrt{2}\left(B + \frac{L^2}{2}\right)c_{00}, \quad c_{02} = \sqrt{2}\left(A + \frac{D^2}{2}\right)c_{00}.$$

Given formal expressions for the objects to be averaged it is necessary to reduce the averaging procedure to a form convenient for the subsequent analytical or numerical treatment. The following sections are devoted to the solution of this problem in different situations, which may occur when considering the general problem (1.1)-(1.6).

3 Average transition probabilities in case of $\epsilon_1 = 0$

If $\epsilon_1 = 0$ the function $\xi(t)$ from (2.11) is nonrandom. We denote it as $\xi_0(t)$, and get

$$\ddot{\xi}_0 + \Omega_0^2(t)\xi_0 = 0, \quad \xi_0(t) \underset{t\to-\infty}{\sim} e^{i\Omega_{in}t}, \tag{3.20}$$

$$\xi_0(t) = r_0(t)e^{\gamma_0(t)} = \xi_{01}(t) + i\xi_{02}(t).$$

We also introduce the designation $\eta_0(t)$ for a function satisfying the equation

$$\ddot{\eta}_0 + \Omega_0^2(t)\eta_0 = F_0(t), \quad \eta_0(-\infty) = \dot{\eta}_0(-\infty) = 0 \tag{3.21}$$

Theorem 3.1. *For any quantity $G(\eta(t), \dot{\eta}(t))$ local with respect to $\eta(t)$ and $\dot{\eta}(t)$ (such are the coefficients c_{nm} in case of $\epsilon_1 = 0$) the averaging formula has the following form*

$$\langle G(\eta(t), \dot{\eta}(t))\rangle = \int\limits_{-\infty}^{+\infty}\!\!\int dx_1 dx_2 \; G(x_1, x_2) P_1(x_1, x_2, t | \eta_0(t_2), \dot{\eta}_0(t_2), t_2), \quad t > t_2. \tag{3.22}$$

where

$$P_1 = \frac{(4b_1 b_3 - b_2^2)^{-1/2}}{2\pi\Omega_{in}} exp\left\{-\frac{b_3 y_1^2 + b_1 y_2^2 - b_2 y_1 y_2}{4b_1 b_3 - b_2^2}\right\}, \tag{3.23}$$

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and

$$b_1(t) = \frac{\epsilon_2}{\Omega_{in}^2} \int\limits_{t_2}^{t} p_2(t') \xi_{01}^2(t') dt', \quad b_3(t) = \frac{\epsilon_2}{\Omega_{in}^2} \int\limits_{t_2}^{t} p_2(t') \left(\dot{\xi}_{01}(t')\right)^2 dt',$$

$$b_2(t) = \frac{2\epsilon_2}{\Omega_{in}^2} \int\limits_{t_2}^{t} p_2(t') \xi_{01}(t') \dot{\xi}_{01}(t') dt',$$

$$\begin{cases} y_1 = -\left[\dot{\xi}_{01}(t)\left(x_1 - \eta_0(t)\right) - \xi_{01}(t)\left(x_2 - \dot{\eta}_0(t)\right)\right] / \Omega_{in}, \\ y_2 = \left[\dot{\xi}_{02}(t)\left(x_1 - \eta_0(t)\right) - \xi_{02}(t)\left(x_2 - \dot{\eta}_0(t)\right)\right] / \Omega_{in}; \end{cases}$$

Proof. It is obvious that the function $\eta(t)$ is nonrandom in the time interval $t < t_2$: $\eta(t) = \eta_0(t)$, while in the interval $t > t_2$ it represents a random process with the evolution governed by the equation

$$\ddot{\eta} + \Omega_0^2(t)\eta = F_0(t) + \sqrt{2\epsilon_2 p_2} f_2(t). \tag{3.24}$$

The initial condition for (3.24) is defined by the requirement for the trajectory and its first derivative that they be continuous at the moment t_2, i.e. $\eta(t_2) = \eta_0(t_2)$, $\dot{\eta}(t_2) = \dot{\eta}_0(t_2)$. To make the analysis of equation (3.24), containing random processes, correct it is convenient to rewrite it as the set of two first order differential equations with respect to the quantities $x_1 = \eta$, $x_2 = \dot{\eta}$

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = F_0 - \Omega_0^2 x_1 + \sqrt{2\epsilon_2 p_2} f_2, \end{cases} \qquad \begin{cases} x_1(t_2) = \eta_0(t_2), \\ x_2(t_2) = \dot{\eta}_0(t_2), \end{cases} \tag{3.25}$$

Equations (3.25) are naturally interpreted as SDE for the random processes $x_1(t)$ and $x_2(t)$. Proceeding from them it is not difficult to write down the Fokker-Planck equation for the conditional probability density

$$P_1(x_1, x_2, t | x_{10}, x_{20}, t_0) = \left\langle \delta(x_1(t) - x_1)\delta(x_2(t) - x_2) \right\rangle \Big|_{\substack{x_1(t_0) = x_{10} \\ x_2(t_0) = x_{20},}}$$

describing the probability that trajectory $(x_1(t), x_2(t))$ finds itself in the vicinity of the point (x_1, x_2) at the moment t, having started from the point (x_{10}, x_{20}) at the moment t_0. It can be shown that the equation for P_1 has the form (see [10] or [11])

$$\frac{\partial P_1}{\partial t} = -x_2 \frac{\partial P_1}{\partial x_1} - (F_0 - \Omega_0^2 x_1)\frac{\partial P_1}{\partial x_2} + \epsilon_2 p_2 \frac{\partial^2 P_1}{\partial x_2^2}. \tag{3.26}$$

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The solution of (3.26) must be integrable and satisfy the obvious initial condition

$$P_1\big|_{t=t_0} = \delta(x_1 - x_{10})\delta(x_2 - x_{20}) \tag{3.27}$$

As the position of the trajectory at the moment t_2 is known, it is natural to set $t_0 = t_2$, $x_{10} = \eta_0(t_2)$, $x_{20} = \dot{\eta}_0(t_2)$. The integrable solution of (3.26), satisfying (3.27), may be expressed in terms of the functions $\xi_{01}(t)$, $\xi_{02}(t)$, $\eta_0(t)$. It is easy to test that it has the form (3.23) by the explicit verification. The theorem is proved. $\triangle$

To find probabilities W_{nm} in this case we calculate the Gaussian integrals and then compute their limiting values at $t \to +\infty$.

4 Average transition probabilities in case of $\epsilon_2 = 0$

In the case of $\epsilon_2 = 0$ the time axis is broken into two parts. At $t < t_1$ functions $\eta(t)$ and $\xi(t)$ are nonrandom: $\eta(t) = \eta_0(t)$, $\xi(t) = \xi_0(t)$. At $t > t_1$ both $\eta(t)$'s and $\xi(t)$'s trajectories become random.

Theorem 4.1. *In the case of $\epsilon_2 = 0$, at $t > t_1$ the set of equations (2.9), (2.11) gives rise to the set of SDE, describing the evolution of four random processes*

$$\vec{u}(t) \equiv (u_1(t), u_2(t), u_3(t), u_4(t)) \equiv \left(\eta(t), \dot{\eta}(t), Re\left(\frac{\dot{\xi}(t)}{\xi(t)}\right), Im\left(\frac{\dot{\xi}(t)}{\xi(t)}\right) \right)$$

with the joint probability distribution function $P_2(\vec{u}, t|\vec{u}_0, t_1)$, $t > t_1$, satisfying the Fokker-Planck equation

$$\frac{\partial P_2}{\partial t} = \hat{L}_2 P_2, \tag{4.28}$$

$$\hat{L}_2(\vec{u}) \equiv -\sum_{i=1}^{4} K_i \frac{\partial}{\partial u_i} + \epsilon_1 p_1 u_1^2 \frac{\partial^2}{\partial u_2^2} + \epsilon_1 p_1 \frac{\partial^2}{\partial u_3^2} + 2\epsilon_1 p_1 u_1 \frac{\partial^2}{\partial u_2 \partial u_3} + 4u_3,$$

$$K_1 = u_2, \quad K_2 = F_0 - \Omega_0^2 u_1, \quad K_3 = u_4^2 - u_3^2 - \Omega_0^2, \quad K_4 = -2u_3 u_4.$$

and the initial condition

$$P_2\big|_{t=t_1} = \delta(\vec{u} - \vec{u}_0) \equiv \prod_{i=1}^{4} \delta(u_i - u_{0i}).$$

Proof. In the case under consideration the equation (2.9) is transformed to a set of SDE in the same way as it has been done in the previous section, namely, by

introducing the quantities $u_1 = \eta$, $u_2 = \dot{\eta}$ which reduce (2.9) to (3.25) with the only distinction in the initial condition: $u_1(t_1) = \eta_0(t_1)$, $u_2(t_1) = \dot{\eta}_0(t_1)$.

The equation (2.11) may be reduced to a nonlinear first order differential equation by the substitution

$$\xi(t) = \begin{cases} \xi_0(t), & t < t_1 \\ \xi_0(t_1) \exp\left\{ \int\limits_{t_1}^{t} \Phi(t') dt' \right\}, & t > t_1, \end{cases} \tag{4.29}$$

which gives upon being applied to (2.11) the following SDE for $\Phi(t)$ in the interval $t_1 < t < \infty$

$$\dot{\Phi}(t) + \Phi^2(t) + \Omega_0^2(t) + \sqrt{2\epsilon_1 p_1} f_1 = 0, \quad \Phi(t_1) = \dot{\xi}_0(t_1)/\xi_0(t_1). \tag{4.30}$$

where the second equation expresses a condition which guarantees continuity of the function $\xi(t)$ and its first derivative at $t = t_1$. The function $\Phi(t)$ is a complex-valued random process due to the initial condition. As a result the SDE (4.30) is equivalent to a set of two SDE for real-valued random processes. Namely, introducing real and imaginary parts of $\Phi(t)$

$$\Phi(t) = u_3(t) + i u_4(t),$$

we finally obtain the following set of SDE for the components of the random vector process $\vec{u}$

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = F_0 - \Omega_0^2 u_1 - \sqrt{2\epsilon_1 p_1} u_1 f_1, \\ \dot{u}_3 = -u_3^2 + u_4^2 - \Omega_0^2(t) - \sqrt{2\epsilon_1 p_1} f_1(t), \\ \dot{u}_4 = -2 u_3 u_4, \end{cases} \quad \begin{cases} u_1(t_1) = \eta_0(t_1), \\ u_2(t_1) = \dot{\eta}_0(t_1), \\ u_3(t_1) = Re\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right), \\ u_4(t_1) = Im\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right). \end{cases} \tag{4.31}$$

The pairs of random processes (u_1, u_2) and (u_3, u_4) are not independent, because their evolution is influenced by the common random force $f_1(t)$. This means that the joint probability distribution

$$P_2(\vec{u}, t | \vec{u}_0, t_1) = \left\langle \prod_{i=1}^{4} \delta(u_i(t) - u_i) \right\rangle \bigg|_{\vec{u}(t_1) = \vec{u}_0}$$

$$\vec{u}_0 = \left(\eta_0(t_1), \dot{\eta}_0(t_1), Re\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right), Im\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right) \right)$$

is a nonfactorable function. Proceeding from the known evolution equations (4.31), we obtain by the standard method the Fokker-Planck equation for P_2 (see [10] or [11]), which has the form (4.28). The theorem is proved. $\triangle$

Given P_2, one can average any quantity $G\left(\vec{u}(t)\right)$ which is local with respect to $\vec{u}(t)$:

$$\langle G\left(\vec{u}(t)\right)\rangle = \int d\vec{u}P_2\left(\vec{u},t|\vec{u}_0,t_1\right)G\left(\vec{u}\right), \quad d\vec{u} = du_1 du_2 du_3 du_4. \tag{4.32}$$

But this formula fails to give a result if it is used for averaging the objects containing the coefficients c_{nm} from (2.17), which are nonlocal with respect to $\vec{u}$. There does not exist a general approach to calculating the average value of any quantity nonlocal with respect to a random process. But it is known that for some types of such objects the averaging procedure may be reduced to finding a fundamental solution of some parabolic partial differential equation and its subsequent weighted integration. The description of the simplest case of this kind is given in [12]. It is not difficult to generalize the result obtained in [12], and therefore the formulas (A.4)-(A.5) can be derived (see Appendix). Using (A.4)-(A.5), we have the following proposition.

Proposition 4.1. *If the components of the random vector process $\vec{u}$ satisfy the set of SDE (4.31), then the averaging procedure can be represented as*

$$\left\langle \exp\left\{-\int_{t_1}^t V_1\left(\vec{u}(\tau),\vec{u}(t),t\right)d\tau - V_2\left(\vec{u}(t)\right)\right\}\right\rangle = \int d\vec{u}e^{-V_2(\vec{u})}Q\left(\vec{u},\vec{u},t\right), \tag{4.33}$$

where the function $Q\left(\vec{u},\vec{u}',t\right)$ is a solution of the problem

$$\frac{\partial Q}{\partial t} = \left(\hat{L}_2\left(\vec{u}\right) - V_1\left(\vec{u},\vec{u}',t\right)\right)Q,$$

$$Q\left(\vec{u},\vec{u}',t\right) \xrightarrow[t\to t_1]{} \delta\left(\vec{u}-\vec{u}_0\right), \quad Q\left(\vec{u},\vec{u}',t\right) \xrightarrow[||\vec{u}||\to\infty]{} 0, \tag{4.34}$$

where $||\cdot||$ is a norm in R^4.

It is not difficult to show that when $V_1 = 0$, the formula (4.33) transforms into (4.32) with the substitution $G = \exp\left\{-V_2\right\}$, because in this case the equation (4.34) for Q transforms into the Fokker-Planck equation (4.28) for P_2.

Using the proposition 4.1, we obtain the representation for the average values $\langle|c_{nm}|^2\rangle$, which are equal to the probabilities W_{nm} in the limit $t \to +\infty$. The explicit expressions for the first four of them are presented here ($n = 0, 1; \ m = 0, 1$) :

$$\left\langle |c_{nm}|^2\right\rangle = \int d\vec{u}H_{nm}\left(\vec{u}\right)Q_{nm}\left(\vec{u},t\right), \tag{4.35}$$

where

$$H_{00}\left(\vec{u}\right) = \frac{2\sqrt{\Omega_{in}\Omega_{out}}}{|\xi_0(t_1)K|}\exp\left\{\frac{\Omega_{out}}{2}\left(\frac{2\Omega_{out}(u_4+\Omega_{out})}{|K|^2}-2\right)u_1^2-\right.$$

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$$-\frac{u_4 + \Omega_{out}}{|K|^2}u_2^2 + \frac{2\Omega_{out}u_3}{|K|^2}u_1u_2\Big\},$$

$$H_{01}(\vec{u}) = \frac{2\Omega_{out}}{|K|^2}\left((u_2 - u_1u_3)^2 + u_1^2u_4^2\right)H_{00}(\vec{u}),$$

$$H_{10}(\vec{u}) = \frac{2\Omega_{in}}{|\xi_0(t_1)K|^2}\left(\Omega_{out}^2u_1^2 + u_2^2\right)H_{00}(\vec{u}), \tag{4.36}$$

$$H_{11}(\vec{u}) = \frac{4\Omega_{in}\Omega_{out}}{|\xi_0(t_1)K^2|^2}\left|K - (\Omega_{out}u_1 - iu_2)(u_1u_4 + i(u_2 - u_1u_3))\right|^2 H_{00}(\vec{u})$$

$$K = \Omega_{out} + u_4 - iu_3,$$

and functions $Q_{nm}(\vec{u}, t)$ satisfy the equations

$$\frac{\partial Q_{nm}}{\partial t} = \hat{L}_2 Q_{nm} - V_{nm}Q_{nm},$$

$$Q_{nm}(\vec{u}, t) \xrightarrow[t \to t_1]{} \delta(\vec{u} - \vec{u}_0), \quad Q_{nm}(\vec{u}, t) \xrightarrow[||\vec{u}|| \to \infty]{} 0, \tag{4.37}$$

where

$$V_{nm} = p_{nm}u_3, \quad p_{00} = p_{01} = 1, \quad p_{10} = p_{11} = 3$$

To obtain W_{nm} it is necessary to proceed in (4.35) to the limit $t \to +\infty$. The representation (4.35)-(4.37) is exact and free from any simplifying assumptions. Given a specific realization of $p_1(t)$, (4.35)-(4.37) is used as a basis for numerical calculations of the probabilities W_{nm}.

More simple representation may be obtained under some additional assumptions. First we introduce some useful designations. Namely, we shall start with the expression for the solution of equation (3.20) applicable at $t \to +\infty$:

$$\xi_0(t) = C_1 e^{i\Omega_{out}t} + C_2 e^{-i\Omega_{out}t}, \quad C_1 = |C_1|e^{i\delta_1}, \quad C_2 = |C_2|e^{i\delta_2},$$

then we shall write down also a corresponding representation for the solution of the equation (3.21):

$$\eta_0(t) = \frac{1}{\sqrt{2\Omega_{in}}}\left(\xi_0 d^* + \xi_0^* d\right), \quad d(t) = \frac{i}{\sqrt{2\Omega_{in}}}\int_{-\infty}^{t}\xi_0(t')F_0(t')dt'.$$

The following quantities, defined on the basis of the above formulas, will be included into the final expressions for the required probabilities

$$\rho = \left|\frac{C_2}{C_1}\right|^2, \quad d = \lim_{t \to +\infty}d(t) = \sqrt{\nu}e^{i\beta}, \tag{4.38}$$

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Theorem 4.2. *Let the function $p_1(t)$ have the form*

$$p_1(t) = \begin{cases} 1 & t_1 < t < t_e, \\ 0 & t > t_e. \end{cases} \qquad (4.39)$$

and time t_e be large enough to guarantee that the solution $Q_{nm}(\vec{u}, t)$ of the equation (4.37) at the moment t_e may be replaced approximately by its stationary (at $t \to +\infty$) limit. Then we can obtain the following representation of the probabilities W_{nm}

$$W_{nm} = \Omega_{in}^{p_{nm}} \int d\xi_1 d\xi_2 d\xi_3 \bar{Q}_{nm}^{st}(\xi_1, \xi_2, \xi_3) \bar{H}_{nm}(\xi_1, \xi_2, \xi_3). \qquad (4.40)$$

where $\bar{Q}_{nm}^{st}$ is a solution of the shortened stationary equation

$$\left(\hat{\bar{L}}_2^{st}(\vec{\xi}) - p_{nm}\xi_3 \right) \bar{Q}_{nm}^{st}(\vec{\xi}) = 0, \qquad (4.41)$$

$$\hat{\bar{L}}_2^{st}(\vec{\xi}) \equiv -\sum_{i=1}^3 \bar{K}_i \frac{\partial}{\partial \xi_i} + \epsilon_1 \xi_1^2 \frac{\partial^2}{\partial \xi_2^2} + \epsilon_1 \frac{\partial^2}{\partial \xi_3^2} + 2\epsilon_1 \xi_1 \frac{\partial^2}{\partial \xi_2 \partial \xi_3} + 2\xi_3.$$

$$\bar{K}_1 = \xi_2, \quad \bar{K}_2 = -\Omega_{out}^2 \xi_1, \quad \bar{K}_3 = -(\xi_3^2 + \Omega_{out}^2).$$

Quantities $\bar{H}_{nm}$ for $n = 0, 1$ and $m = 0$ are given by

$$\bar{H}_{00}(\xi_1, \xi_2, \xi_3) = \frac{2\sqrt{\Omega_{in}\Omega_{out}}}{|\xi_0(t_1)|\sqrt{\Sigma(\xi_3)}} \exp \left\{ -\frac{\Omega_{out}\Omega_{in}^2}{\Sigma(\xi_3)} [\xi_3(\xi_1 + \mu_1) - \xi_2 - \mu_2]^2 \right\},$$

$$\bar{H}_{01}(\xi_1, \xi_2, \xi_3) = \frac{2\Omega_{out}\Omega_{in}^2}{\Sigma(\xi_3)} [\xi_2 - \xi_1\xi_3 - \xi_3\mu_1 + \mu_2]^2 \bar{H}_{00}(\xi_1, \xi_2, \xi_3),$$

$$\mu_1 = -d_5 + \sqrt{\frac{2\nu}{\Omega_{in}}}(d_1 \cos\beta + d_2 \sin\beta)$$

$$\mu_2 = -d_6 + \sqrt{\frac{2\nu}{\Omega_{in}}}(d_3 \cos\beta + d_4 \sin\beta) \qquad (4.42)$$

$$\Sigma(\xi_3) = \frac{\Omega_{in}\Omega_{out}}{(1-\rho)} \left\{ \left[(d_1^2 + d_2^2)(1+\rho) - 2\sqrt{\rho} \left[(d_1^2 - d_2^2) \cos\delta + 2d_1 d_2 \sin\delta \right] \right] \xi_3^2 + \right.$$

$$\left. +2 \left[-(d_2 d_4 + d_1 d_3)(1+\rho) + 2\sqrt{\rho} \left[(d_1 d_4 + d_2 d_3) \sin\delta + (d_1 d_3 - d_2 d_4) \cos\delta \right] \right] \xi_3 + \right.$$

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$$+\left[(d_3^2 + d_4^2)(1 + \rho) + 2\sqrt{\rho}\left[(d_4^2 - d_3^2)\cos\delta - 2d_3d_4\sin\delta\right]\right]\right\}.$$

where $\delta = \delta_1 + \delta_2$. The ambiguity which arises in finding the solution $\bar{Q}_{nm}^{st}$ of the equation (4.41) is eliminated by the requirement of the correspondence of $\bar{Q}_{nm}^{st}$ to the stationary limit of the solution Q_{nm} of the equation (4.37).

Proof. The proof originates from the representation (4.35)-(4.37). Denote the solution of the problem (4.37) in the time interval $t_1 < t < t_e$ by $Q_{nm}^<(\vec{u}, t)$ and that in the time interval $t > t_e$ by $Q_{nm}^>(\vec{u}, t)$. Then $Q_{nm}^>(\vec{u}, t)$ satisfies the first order differential equation

$$\frac{\partial Q_{nm}^>}{\partial t} = -\sum_{i=1}^4 \frac{\partial(K_i Q_{nm}^>)}{\partial u_i} - V_{nm}Q_{nm}^>,$$

$$Q_{nm}^>(\vec{u}, t) \xrightarrow[t \to t_e]{} Q_{nm}^<(\vec{u}, t_e), \qquad Q_{nm}^>(\vec{u}, t) \xrightarrow[||\vec{u}|| \to \infty]{} 0. \tag{4.43}$$

Using the method of characteristics the problem (4.43) can be solved for an arbitrary initial condition. So that we obtain the following expression containing the solutions of classical equations of motion (3.20) and (3.21):

$$Q_{nm}^>(\vec{u}, t) = Q_{nm}^<\left(\vec{\xi}, t_e\right) \cdot \exp\left((4 - p_{nm})\int_{t_e}^t u_3\left(\vec{\xi}, \tau\right) d\tau\right), \tag{4.44}$$

Quantities $\xi_i(\vec{u}, t)$, $i = 1, .., 4$ in (4.44) are the first integrals of the characteristic set of ordinary first order differential equations, corresponding to the equation (4.39). Their dependency on $\vec{u}$ variables is given implicitly by the relations

$$u_1 = \frac{1}{\Omega_{in}}[e_{22}\xi_1 + e_{21}\xi_2 + h_1], \qquad u_2 = \frac{1}{\Omega_{in}}[e_{12}\xi_1 + e_{11}\xi_2 + h_2],$$

$$u_3 = \frac{(e_{11}\xi_3 + e_{12})(e_{21}\xi_3 + e_{22}) + e_{11}e_{21}\xi_4^2}{(e_{21}\xi_3 + e_{22})^2 + e_{21}^2\xi_4^2}, \quad u_4 = \frac{\Omega_{in}\xi_4}{(e_{21}\xi_3 + e_{22})^2 + e_{21}^2\xi_4^2}. \tag{4.45}$$

The definitions of functions and constants appearing in (4.45) are given by

$$h_1(t) = (d_2d_6 - d_4d_5)\xi_{01}(t) + (d_3d_5 - d_1d_6)\xi_{02}(t) + \Omega_{in}\eta_0(t), \quad h_2(t) = \dot{h}_1(t);$$

$$\begin{aligned}
e_{21}(t) &= d_1\xi_{02}(t) - d_2\xi_{01}(t), & e_{12}(t) &= \dot{e}_{22}(t), \\
e_{22}(t) &= d_4\xi_{01}(t) - d_3\xi_{02}(t), & e_{11}(t) &= \dot{e}_{21}(t); \\
d_1 &= \xi_{01}(t_e), \; d_2 = \xi_{02}(t_e), & d_5 &= \eta_0(t_e), \\
d_3 &= \dot{\xi}_{01}(t_e), \; d_4 = \dot{\xi}_{02}(t_e), & d_6 &= \dot{\eta}_0(t_e).
\end{aligned} \tag{4.46}$$

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The functions $\xi_{01}(t)$, $\xi_{02}(t)$ and $\eta_0(t)$ are defined in (3.20) and (3.21). By the explicit verification it may be proved that the new variables satisfy the initial conditions

$$\xi_i\left(\vec{u}, t_e\right) = u_i, \quad i = 1, .., 4, \tag{4.47}$$

so that the solution (4.44) satisfies the initial condition defined in (4.43).

Using (4.45), the Jacobian

$$J\left(\vec{\xi}\right) \equiv \frac{\partial(u_1, .., u_4)}{\partial(\xi_1, .., \xi_4)} \tag{4.48}$$

of transition to new variables is calculated. In case of $\xi_4 = 0$, which is important in what follows, it takes form

$$J\left(\vec{\xi}\right)\Big|_{\xi_4=0} = \frac{\Omega_{in}^4}{(e_{21}\xi_3 + e_{22})^4} \tag{4.49}$$

Proceeding from the above we can rewrite the representation (4.35) as:

$$\left\langle |c_{nm}|^2 \right\rangle = \int d\vec{\xi}\, J\left(\vec{\xi}\right) H_{nm}\left(\vec{u}(\vec{\xi}, t)\right) Q_{nm}^<\left(\vec{\xi}, t_e\right) \exp\left((4 - p_{nm}) \int_{t_e}^{t} u_3\left(\vec{\xi}, \tau\right) d\tau \right), \tag{4.50}$$

where integration over τ is carried out under the assumption that $\vec{\xi}$ is constant. Assuming that the moment t_e is big enough the solution $Q_{nm}^<\left(\vec{\xi}, t_e\right)$ can be replaced approximately by the stationary limit $Q_{nm}^{st}\left(\vec{\xi}\right)$, which satisfies the equation

$$\left(\hat{L}_2^{st} - V_{nm}\right) Q_{nm}^{st} = 0, \quad \hat{L}_2^{st} = \lim_{t\to+\infty} \hat{L}_2 \tag{4.51}$$

Operator $\hat{L}_2$ is defined in (4.28). If necessary an exact value of t_e, which will be sufficient for a legality of the replacement mentioned above, may be estimated. It is obvious that the stationary limit $Q_{nm}^{st}(\vec{u})$ of the function $Q_{nm}(\vec{u}, t)$ depends on the initial condition for the latter.

The insertion of the stationary solution Q_{nm}^{st} into the formulas make them more simple, because the stationary equation allows a separation of variables. So substituting

$$Q_{nm}^{st} = \delta(\xi_4)\bar{Q}_{nm}^{st}(\xi_1, \xi_2, \xi_3), \tag{4.52}$$

into (4.51), we obtain the following (shortened) equation determining the function $\bar{Q}_{nm}^{st}$

$$\left(\hat{\bar{L}}_2^{st}\left(\vec{\xi}\right) - p_{nm}\xi_3\right) \bar{Q}_{nm}^{st}(\vec{\xi}) = 0, \tag{4.53}$$

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$$\hat{\bar{L}}_2^{st}\left(\vec{\xi}\right) \equiv -\sum_{i=1}^{3} \bar{K}_i \frac{\partial}{\partial \xi_i} + \epsilon_1 \xi_1^2 \frac{\partial^2}{\partial \xi_2^2} + \epsilon_1 \frac{\partial^2}{\partial \xi_3^2} + 2\epsilon_1 \xi_1 \frac{\partial^2}{\partial \xi_2 \partial \xi_3} + 2\xi_3.$$

$$\bar{K}_1 = \xi_2, \quad \bar{K}_2 = -\Omega_{out}^2 \xi_1, \quad \bar{K}_3 = -(\xi_3^2 + \Omega_{out}^2).$$

Substituting (4.52) into (4.50) and using (4.49), we come to the formula

$$\left\langle |c_{nm}|^2 \right\rangle = \Omega_{in}^{p_{nm}} \int d\xi_1 d\xi_2 d\xi_3 \frac{\bar{Q}_{nm}^{st}(\xi_1, \xi_2, \xi_3)}{(e_{21}\xi_3 + e_{22})^{p_{nm}}} H_{nm}\left(\vec{u}(\vec{\xi}, t)\right) |_{\xi_4=0}. \tag{4.54}$$

To find the average probabilities W_{nm} we proceed to the limit $t \to +\infty$ in (4.54). This may be done for any n and m. After implementing simple but wearisome transformations the limiting values of (4.54) were obtained for the minimal n, m. They have the form (4.40)-(4.42). The theorem is proved. $\triangle$

It is a real challenge to obtain the solution $\bar{Q}_{nm}^{st}$, corresponding to a given initial condition appearing in (4.37), without solving the equation on the whole interval $t_1 < t < \infty$. But this problem may be resolved in the following particular case.

Theorem 4.3. *Let $F_0(t) \equiv const = F_0$, $\Omega_0(t) \equiv const = \Omega_0$, then $\bar{Q}_{nm}^{st}$ entering into the statement of the theorem 4.2 is given by*

$$\bar{Q}_{nm}^{st}(u_1, u_2, u_3) = C_{nm} \bar{q}_{nm}^{st}(u_1, u_2, u_3), \tag{4.55}$$

where $\bar{q}_{nm}^{st}(u_1, u_2, u_3)$ is an arbitrary fixed solution of (4.53) decreasing on infinity, and the constant C_{nm} is expressed in terms of this solution as follows

$$C_{nm} = Y_{nm}^{st}(\vec{u}_0) \left\{ \int du_1 du_2 du_3 \, Y_{nm}^{st}(\vec{u})\Big|_{u_4=0} \bar{q}_{nm}^{st}(u_1, u_2, u_3) \right\}^{-1} \tag{4.56}$$

where Y_{nm}^{st} is any solution of the conjugate equation

$$\left(\left(\hat{L}_2^{st}\right)^+ (\vec{u}) - p_{nm} u_3 \right) Y_{nm}^{st}(\vec{u}) = 0, \tag{4.57}$$

$$\left(\hat{\bar{L}}_2^{st}\right)^+ (\vec{u}) \equiv \sum_{i=1}^{4} K_i^{st} \frac{\partial}{\partial u_i} + \epsilon_1 u_1^2 \frac{\partial^2}{\partial u_2^2} + \epsilon_1 \frac{\partial^2}{\partial u_3^2} + 2\epsilon_1 u_1 \frac{\partial^2}{\partial u_2 \partial u_3}.$$

$$K_1^{st} = u_2, \quad K_2^{st} = -\Omega_{out}^2 u_1, \quad K_3^{st} = u_4^2 - u_3^2 - \Omega_{out}^2), \quad K_4^{st} = -2u_3 u_4.$$

Proof. To prove the above statement we consider the quantity

$$I_{nm}^{(Y)}(t) \equiv \int d\vec{u} \, Y_{nm}(\vec{u}, t) Q_{nm}(\vec{u}, t), \tag{4.58}$$

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where $Q_{nm}(\vec{u}, t)$ is a solution of the problem (4.37) and function $Y_{nm}(\vec{u}, t)$ is assumed to be found proceeding from the requirement that (4.58) does not depend on the time variable, i.e. $dI_{nm}^{(Y)}/dt = 0$. Integrating by parts we can easily show that the requirement will be satisfied if the function Y_{nm} is an arbitrary solution of the equation conjugate to (4.37)

$$\frac{\partial Y_{nm}}{\partial t} = \left(-\hat{L}_2^+ + V_{nm}\right) Y_{nm}, \qquad (4.59)$$

$$\hat{L}_2^+(\vec{u}) \equiv \sum_{i=1}^4 K_i \frac{\partial}{\partial u_i} + \epsilon_1 p_1 u_1^2 \frac{\partial^2}{\partial u_2^2} + \epsilon_1 p_1 \frac{\partial^2}{\partial u_3^2} + 2\epsilon_1 p_1 u_1 \frac{\partial^2}{\partial u_2 \partial u_3}.$$

The only condition which is supposed to be imposed on Y_{nm} is its sufficiently rapid decreasing with $||\vec{u}|| \to \infty$. This guarantees that all the terms, containing integration over infinitely remote surfaces, will vanish.

It is worth mentioning that the quantity (4.58) does not depend on time even when the functions $\Omega_0(t)$ and $F_0(t)$ entering into the statement of the theorem are nonconstant. This condition becomes important in the case when the constancy of the quantity (4.58) is used for the restoration of information about the initial condition, which is lost in proceeding to the stationary limit in (4.37).

If $p_1(t) = p_2(t) \equiv 1$, $F_0(t) \equiv const = F_0$, $\Omega_0(t) \equiv const = \Omega_0$ the operator $\hat{L}_2^+$ which is conjugate to the Fokker-Planck operator $\hat{L}_2$ does not depend on time. In this case it may be the time independent solution Y_{nm}^{st} of the equation (4.59)

$$\left(-\hat{L}_2^+ + p_{nm} u_3\right) Y_{nm}^{st} = 0. \qquad (4.60)$$

to be substituted into (4.58) instead of Y_{nm}. Note that in this case $\hat{L}_2^+ = \left(\hat{L}_2^{st}\right)^+$, so that the equation (4.60) is in fact a stationary equation corresponding to (4.59). Having chosen such a function Y_{nm} one can equate values of the quantity (4.58) at $t = t_1$ and at $t \to \infty$:

$$Y_{nm}^{st}(\vec{u}_0) = \int d\vec{u} Y_{nm}^{st}(\vec{u}) Q_{nm}^{st}(\vec{u}). \qquad (4.61)$$

Proceeding from an arbitrary solution $q_{nm}^{st}(\vec{u})$ of the stationary equation (4.51) with a sufficiently rapid decreasing on infinity we obtain the required solution $Q_{nm}^{st}(\vec{u})$

$$Q_{nm}^{st}(\vec{u}) = C_{nm} q_{nm}^{st}(\vec{u}), \qquad (4.62)$$

where the constant C_{nm} depends on the specific choice of $q_{nm}^{st}(\vec{u})$. Unfortunately, so far we do not possess a strict proof of such representation, but it seems plausible in

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view of the results obtained in exploring the more simple one-dimensional equation. Substituting (4.62) into (4.58), one readily obtains

$$C_{nm} = \frac{Y_{nm}^{st}(\vec{u}_0)}{\int d\vec{u}\, Y_{nm}^{st}(\vec{u}) q_{nm}^{st}(\vec{u})}. \tag{4.63}$$

Taking into account that variables in stationary equations are separated, i.e. $Q_{nm}^{st}(\vec{u}) = \delta(u_4)\bar{Q}_{nm}^{st}(u_1, u_2, u_3)$ $q_{nm}^{st}(\vec{u}) = \delta(u_4)\bar{q}_{nm}^{st}(u_1, u_2, u_3)$, the formula (4.56) for C_{nm} is obtained. $\triangle$

5 Average transition probabilities in case of $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$

If influence of both random forces is taken into account, i.e. $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$, evolution of the system depends on the correlation of the moments t_1 and t_2. If $t < t_> \equiv max(t_1, t_2)$ equations (2.9) and (2.11) for the trajectories $\eta(t)$ and $\xi(t)$ are different, depending on whether $t_> = t_1$ or $t_> = t_2$. The two possibilities correspond to the two probability distributions of random variables $\eta(t_>)$ and $\xi(t_>)$, which are the initial values for the corresponding trajectories on the interval $t > t_>$. Denote $z_1 = \eta$, $z_2 = \dot{\eta}$, $z_3 = Re(\dot{\xi}/\xi)$, $z_4 = Im(\dot{\xi}/\xi)$. For the probability density function $R(\vec{z}, t)$, which is defined as a probability for the trajectory $\vec{z}(t)$ to be found in the interval $[\vec{z},\ \vec{z} + d\vec{z}]$ at the moment $t > t_< \equiv min(t_1, t_2)$, we can write down the following expression

$$R(\vec{z}, t_>) = \begin{cases} P_1(z_1, z_2, t_> | z_{01}, z_{02}, t_<)\delta(z_3 - z_{03})\delta(z_4 - z_{04}), & t_> = t_1, \\[2mm] P_2(\vec{z}, t_> | \vec{z}_0, t_<), & t_> = t_2, \end{cases} \tag{5.64}$$

where

$$\vec{z}_0 = (z_{01}, z_{02}, z_{03}, z_{04}) = \left(\eta_0(t_<),\ \dot{\eta}_0(t_<),\ Re(\dot{\xi}_0(t_<)/\xi_0(t_<)),\ Im(\dot{\xi}_0(t_<)/\xi_0(t_<))\right).$$

and functions P_1 and P_2 are the solutions of (3.26) and (4.28), respectively. It is obvious that the normalization condition $\int R(\vec{z}, t_>)d\vec{z} = 1$ holds.

At $t > t_>$ we have the same set of SDE for the components of the random vector process $\vec{z}(t)$ both at $t_> = t_1$ and $t_> = t_2$. Its inference literally reproduces the

derivation of the set (4.31) and results in

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = F_0 - \Omega_0^2 z_1 - \sqrt{2\epsilon_1 p_1} z_1 f_1 + \sqrt{2\epsilon_2 p_2} f_2, \\ \dot{z}_4 = -2z_3 z_4, \\ \dot{z}_3 = z_4^2 - z_3^2 - \Omega_0^2(t) - \sqrt{2\epsilon_1 p_1} f_1(t), \qquad \vec{z}(t_>) = \vec{z}_>, \end{cases} \tag{5.65}$$

where distribution $R(\vec{z}_>, t_>)$ of the components of the random vector $\vec{z}_>$ is given by the formula (5.64). The representation of the joint probability density

$$P_3(\vec{z}, t | \vec{z}_>, t_>) = \left\langle \delta(\vec{z}(t) - \vec{z}) \right\rangle \Big|_{\vec{z}(t_>) = \vec{z}_>},$$

derived by the standard method from (5.65), is given by the Fokker-Planck equation. Thus we arrive at the analogue of the theorem 4.1

Theorem 5.1. *In the case of $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$ the set of equations (2.9), (2.11) generates at $t > t_>$ the set of SDE describing the evolution of four random processes*

$$\vec{z}(t) \equiv (z_1(t), z_2(t), z_3(t), z_4(t)) \equiv \left(\eta(t), \dot{\eta}(t), Re\left(\frac{\dot{\xi}(t)}{\xi(t)} \right), Im\left(\frac{\dot{\xi}(t)}{\xi(t)} \right) \right)$$

with the joint probability distribution $P_3(\vec{z}, t | \vec{z}_>, t_>)$, $t > t_>$ satisfying the Fokker-Planck equation

$$\frac{\partial P_3}{\partial t} = \hat{L}_3 P_3, \tag{5.66}$$

$$\hat{L}_3(\vec{z}) \equiv -\sum_{i=1}^{4} K_i \frac{\partial}{\partial z_i} + (\epsilon_2 p_2 + \epsilon_1 p_1 z_1^2) \frac{\partial^2}{\partial z_2^2} + \epsilon_1 p_1 \frac{\partial^2}{\partial z_3^2} + 2\epsilon_1 p_1 z_1 \frac{\partial^2}{\partial z_2 \partial z_3} + 4z_3,$$

$$K_1 = z_2, \quad K_2 = F_0 - \Omega_0^2 z_1, \quad K_3 = z_4^2 - z_3^2 - \Omega_0^2, \quad K_4 = -2z_3 z_4.$$

and the initial condition

$$P_3 \Big|_{t=t_>} = \delta(\vec{z} - \vec{z}_>),$$

with the probability distribution of components of the random vector $\vec{z}_>$ given by the formula (5.64).

It is not difficult to show that if $\epsilon_1 = 0$ the equation (5.66) transforms into (3.26) by the substitution

$$P_3(\vec{z}, t | \vec{z}_>, t_>) = \delta\left(z_3 - Re\frac{\dot{\xi}_0(t)}{\xi_0(t)} \right) \delta\left(z_4 - Im\frac{\dot{\xi}_0(t)}{\xi_0(t)} \right) P(z_1, z_2, t).$$

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When $\epsilon_2 = 0$ the equation (5.66) transforms directly into (4.28). Thus the relation of the general case with the particular situations, considered in the previous sections, is established.

Using again the formulas (A.4)-(A.5), the representation (4.35) of $\langle |c_{nm}|^2 \rangle$ may be generalized to the case under consideration

$$\langle |c_{nm}|^2 \rangle = \int d\vec{z}_> R(\vec{z}_>, t_>) \int d\vec{z} H_{nm}(\vec{z}) Q_{nm}(\vec{z}, t). \tag{5.67}$$

As compared with (4.35), an additional integration with the weighting function $R(\vec{z}_>, t_>)$ allowing for the dispersion of initial values of the trajectory, was included into (5.67). Functions $H_{nm}(\vec{z})$ are defined in (4.36) and functions $Q_{nm}(\vec{z}, t)$, depending on $z_>$ as on a parameter, are solutions of the following problem

$$\frac{\partial Q_{nm}}{\partial t} = \hat{L}_3 Q_{nm} - V_{nm} Q_{nm},$$

$$Q_{nm}(\vec{z}, t) \xrightarrow[t \to t_>]{} \delta(\vec{z} - \vec{z}_>), \quad Q_{nm}(\vec{z}, t) \xrightarrow[||\vec{z}|| \to \infty]{} 0, \tag{5.68}$$

where

$$V_{nm} = p_{nm} z_3, \quad p_{00} = p_{01} = 1, \quad p_{10} = p_{11} = 3.$$

Proceeding from (5.67)-(5.68) the theorems analogous to the theorems 4.2 and 4.3 can be proved. Here we shall combine them into the one

Theorem 5.2. *Let $\Omega_0(t) \equiv const = \Omega_0$, $F_0(t) \equiv const = F_0$. Let the functions $p_i(t)$ have the form*

$$p_i(t) = \begin{cases} 1 & t_1 < t < t_e, \\ 0 & t > t_e. \end{cases} \tag{5.69}$$

and let the moment t_e lie far enough in the future to provide the legitimacy of the approximate replacement of the solution $Q_{nm}(\vec{u}, t)$ of the equation (4.37) taken at this moment by its stationary limit at $t \to \infty$. Then we have the following representation for the probabilities W_{nm}

$$W_{nm} = \Omega_{in}^{p_{nm}} A_{nm}(t_>) \int d\xi_1 d\xi_2 d\xi_3 \bar{q}_{nm}^{st}(\xi_1, \xi_2, \xi_3) \bar{H}_{nm}(\xi_1, \xi_2, \xi_3). \tag{5.70}$$

$$A_{nm}(t_>) \equiv \int d\vec{z}_> R(\vec{z}_>, t_>) C_{nm}(\vec{z}_>)$$

where $\bar{H}_{nm}$ are defined in (4.42), the function $\bar{q}_{nm}^{st}$ is an arbitrary (decreasing on infinity) solution of the shortened stationary equation

$$\left(\hat{\bar{L}}_3^{st} - p_{nm} \xi_3 \right) \bar{q}_{nm}^{st} = 0, \tag{5.71}$$

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$$\hat{\bar{L}}_3^{st}(\vec{\xi}) \equiv -\sum_{i=1}^{4} \bar{K}_i \frac{\partial}{\partial \xi_i} + (\epsilon_2 + \epsilon_1 \xi_1^2)\frac{\partial^2}{\partial \xi_2^2} + \epsilon_1 \frac{\partial^2}{\partial \xi_3^2} + 2\epsilon_1 \xi_1 \frac{\partial^2}{\partial \xi_2 \partial \xi_3} + 2z_3,$$

$$\bar{K}_1 = z_2, \quad \bar{K}_2 = -\Omega_{out}^2 z_1, \quad \bar{K}_3 = -(z_3^2 + \Omega_{out}^2),$$

and the function $C_{nm}(\vec{z}_>)$ depending on the parameter $z_>$ is defined by the formula

$$C_{nm} = Y_{nm}^{st}(\vec{z}_>) \left\{ \int dz_1 dz_2 dz_3 \; Y_{nm}^{st}(\vec{z})\Big|_{z_4=0} \bar{q}_{nm}^{st}(z_1, z_2, z_3) \right\}^{-1} \tag{5.72}$$

where $Y_{nm}^{st}(\vec{z})$ is an arbitrary solution of the conjugate equation

$$\left(\left(\hat{L}_3^{st}\right)^{+}(\vec{z}) - p_{nm} z_3 \right) Y_{nm}^{st}(\vec{z}) = 0, \tag{5.73}$$

$$\left(\hat{\bar{L}}_3^{st}\right)^{+}(\vec{z}) \equiv \sum_{i=1}^{4} K_i^{st} \frac{\partial}{\partial z_i} + (\epsilon_2 + \epsilon_1 z_1^2)\frac{\partial^2}{\partial z_2^2} + \epsilon_1 \frac{\partial^2}{\partial z_3^2} + 2\epsilon_1 z_1 \frac{\partial^2}{\partial z_2 \partial z_3},$$

$$K_1^{st} = z_2, \quad K_2^{st} = -\Omega_{out}^2 z_1, \quad K_3^{st} = z_4^2 - z_3^2 - \Omega_{out}^2), \quad K_4^{st} = -2z_3 z_4.$$

6 Another approach to the calculation of average transition probabilities

There is an alternative approach to calculation of the average transition probabilities. The main idea of the method will be demonstrated with the particular case of the problem (1.1) - (1.6).

Theorem 6.1. *Let in (1.3) $\Omega_0 = F_0 = 0$, $\epsilon_1 = 0$. Then the solution of the problem (1.1)-(1.3) may be represented as*

$$\Psi_{stc}(t, x) = e^{i(\xi_1(x-\xi_2)+\xi_3)}\phi(x - \xi_2, t) \tag{6.74}$$

where function $\phi(y, t)$ satisfies a usual partial differential equation

$$i\frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \phi}{\partial^2 y} + \frac{i}{2}y^2\phi,$$

and random processes ξ_1, ξ_2, ξ_3 satisfy the system of ordinary SDE

$$\begin{cases} d\xi_1(t) &= -i\xi_2(t)dt + \sqrt{2\epsilon_2}dW_2(t) \\ d\xi_2(t) &= \xi_1(t)dt \\ d\xi_3(t) &= \frac{1}{2}(\xi_1^2(t) - i\xi^2(t))dt + \sqrt{2\epsilon_2}\xi_2(t)dW_2(t). \end{cases} \tag{6.75}$$

Proof. Let us rewrite the equation (1.1) as a SDE [11]

$$id\Psi_{stc} = -\frac{1}{2}\frac{\partial^2\Psi_{stc}}{\partial x^2}dt - \sqrt{2\epsilon_2}\Theta(t-t_2)x\Psi_{stc}dW_2(t), \tag{6.76}$$

which determines an increment of the random process Ψ_{stc} during the time interval dt. The quantity $W_2(t)$ in (6.76) is the Wiener process $\left(W_2(t) = \int_{t_2}^t f_2(t')dt'\right)$, i.e. the Gaussian random process which is completely determined by the initial condition $W_2(t_2) = 0$ and transition probability

$$P(w_0, t_0|w, t) = \frac{1}{\sqrt{2\pi(t-t_0)}}\exp\left[-\frac{(w-w_0)^2}{2(t-t_0)}\right].$$

The equation (6.76) differs from an ordinary SDE in that it includes derivatives of the process with respect to parameter x. The idea of the method is to represent Ψ_{stc} as a function of several random processes satisfying the set of ordinary SDE, the latter being chosen so as (6.76) would be turned into an identity after the substitution of Ψ_{stc}. In doing so it is necessary to apply the Ito calculus as follows. If $\Psi_{stc} = \phi(\vec{\xi})$, where $\vec{\xi} = (\xi_1, ..., \xi_n)$ is a set of random processes satisfying the set of ordinary SDE

$$d\xi_i(t) = a_i(\vec{\xi}, t)dt + b_i(\vec{\xi}, t)dW_2(t), \quad i = 1, ..., n, \tag{6.77}$$

and ϕ is a differentiable function of its arguments, then an increment of the process Ψ_{stc} is

$$d\Psi_{stc}(t) = \left[\sum_{i=1}^n\left(a_i\frac{\partial\phi}{\partial\xi_i} + \frac{1}{2}b_i^2\frac{\partial^2\phi}{\partial^2\xi_i^2}\right) + \sum_{i\neq j}b_ib_j\frac{\partial^2\phi}{\partial\xi_i\partial\xi_j}\right]dt+$$

$$+\sum_{i=1}^n b_i\frac{\partial\phi}{\partial\xi_i}dW_2(t). \tag{6.78}$$

where $d\xi_i$ are infinitely small increments of the processes ξ_i determined from (6.77). Using this rule, we can show after simple but cumbersome manipulations that the equation (6.76) holds at $t > t_2$, if Ψ_{stc} is represented by (6.74)-(6.75). $\triangle$

The initial conditions for ξ_i may be set to zero: $\xi_1(t_2) = \xi_2(t_2) = \xi_3(t_2) = 0$. Then solving the equation (6.76) at $t < t_2$ and denoting the solution as $\Psi_{stc}^<(t, x)$, we obtain the initial condition for ϕ as: $\phi(t_2, x) = \Psi_{stc}^<(t_2, x)$. The boundary conditions for ϕ are imposed by the natural requirement of normability of the solution.

Having the solution (6.74) it is not difficult to find the matrix elements of different quantum mechanical operators and the probabilities of transitions into given stationary states (for example, to ϕ_n^{out}). To be able to average these quantities over the joint distribution $P(t, \vec{\xi})$, we must solve the Fokker-Planck equation corresponding to the set (6.75).

Unfortunately a substitution reducing the equation (1.1) to the set of ordinary SDE is unknown in the general case (1.1)-(1.6). Therefore at present the described method provides noting more than an instructive illustration of the one of possible approaches to the problem to be solved. Though not leading to the final results, it reserves the field for the further study. Undoubtedly it would be very interesting to obtain the solution with the help of the described procedure, even if only to compare it to those found in the previous sections.

7 Thermodynamics within the framework of representation by stochastic density matrix. Thermodynamical characteristics of oscillator

It is well known [13] that the key object of interest in quantum mechanics is the density matrix.

Definition 7.1. *The stochastic density matrix is defined by the expression*

$$\rho_{stc}\left(x,t;\{\vec{\zeta}\}|x',t';\{\vec{\zeta}'\}\right) = \sum_{m=0}^{\infty} w_0^{(m)} \rho_{stc}^{(m)}\left(x,t;\{\vec{\zeta}\}|x',t';\{\vec{\zeta}'\}\right), \qquad (7.79)$$

$$\rho_{stc}^{(m)}\left(x,t;\{\vec{\zeta}\}|x',t';\{\vec{\zeta}'\}\right) = \sqrt{\frac{\pi}{\Omega_{in}}} \Psi_{stc}^{(m)}\left(x,t|\{\vec{\zeta}\}\right) \overline{\Psi_{stc}^{(m)}\left(x',t'|\{\vec{\zeta}'\}\right)}, \qquad (7.80)$$

where $w_0^{(m)}$ has the meaning of the initial distribution over quantum states with energies $E_m = \left(\frac{1}{2} + m\right)\Omega_{in}$, until the moment when the generator of random excitations is activated.

Definition 7.2. *The expected value of the operator $\hat{A}\left(x,t|\{\vec{\zeta}\}\right)$ in quantum state with the index m is*

$$A_m = \lim_{t \to +\infty} \left\{ Sp_x\left[Sp_{\{\vec{\zeta}\}} \hat{A} \rho_{stc}^{(m)}\right] \middle/ Sp_x\left[Sp_{\{\vec{\zeta}\}} \rho_{stc}^{(m)}\right] \right\}. \qquad (7.81)$$

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The mean value of the operator $\hat{A}\left(x, t | \{\vec{\zeta}\}\right)$ over the whole ensemble of states will respectively be given by

$$A = \lim_{t \to +\infty} \left\{ Sp_x \left[Sp_{\{\vec{\zeta}\}} \hat{A} \rho_{stc} \right] \Big/ Sp_x \left[Sp_{\{\vec{\zeta}\}} \rho_{stc} \right] \right\}. \tag{7.82}$$

The operation Sp_x in (7.81) and (7.82) is defined by

$$Sp_x\{K(x, x')\} = \sqrt{\frac{\Omega_{in}}{\pi}} \int dx K(x, x) \tag{7.83}$$

for any function $K(x, x')$.

Using (7.80) and the properties of the functionals $\Psi_{stc}^{(m)}\left(x, t | \{\vec{\zeta}\}\right)$ we easily obtain the expression for the total nonstationary distribution function

$$w_0 = Sp_x Sp_{\{\vec{\zeta}\}} \left\{ \rho_{stc}\left(x, t; \{\vec{\zeta}\} | x', t'; \{\vec{\zeta}'\}\right) \right\} = \sum_{m=0}^{\infty} w_0^{(m)}. \tag{7.84}$$

If the initial weighting functions $w_0^{(m)}$ are given by the canonical distribution $w_0^{(m)} = \exp\left(-E_m/kT\right)$, the expression (7.84) takes the form of the Planck distribution (see [13])

$$w_0\left(\beta\right) = \frac{e^{\beta/2}}{e^{\beta} - 1}, \qquad \beta = \frac{\Omega_{in}}{kT}. \tag{7.85}$$

Substituting the expansion (2.14) of the wave functional in *out*-states into (7.79)-(7.80) we have the following representation

$$\rho_{stc}\left(x, t; \{\vec{\zeta}\} | x', t'; \{\vec{\zeta}'\}\right) =$$

$$= \sum_{m,k,l=0}^{\infty} w_0^{(m)} c_{mk}\left(t | \{\vec{\zeta}\}\right) \overline{c_{ml}\left(t | \{\vec{\zeta}\}\right)} \phi_k^{out}(x, t) \overline{\phi_l^{out}(x', t)} \tag{7.86}$$

Definition 7.3. *The nonequilibrium partial distribution function is defined by*

$$w^{(m)}\left(\epsilon_1, \epsilon_2, t\right) = Sp_{\{\vec{\zeta}\}} \left\{ \sum_{k=0}^{\infty} \left[w_0^{(k)} \left| c_{km}\left(t | \{\vec{\zeta}\}\right) \right|^2 - w_0^{(m)} \left| c_{mk}\left(t | \{\vec{\zeta}\}\right) \right|^2 \right] \right\} + w_0^{(m)} =$$

$$= \sum_{k=0}^{\infty} \left\{ \left[w_0^{(k)} \Delta_{km}\left(t\right) - w_0^{(m)} \Delta_{mk}\left(t\right) \right] \right\} + w_0^{(m)}, \tag{7.87}$$

where

$$\Delta_{km}\left(t\right) = Sp_{\{\vec{\zeta}\}} \left| c_{km}\left(t | \{\vec{\zeta}\}\right) \right|^2 = \left\langle \left| c_{mm}\left(t | \{\vec{\zeta}\}\right) \right|^2 \right\rangle.$$

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In this case the total distribution function is equal to the sum

$$w_0 = \sum_{m=0}^{\infty} w^{(m)}\left(\epsilon_1, \epsilon_2, t\right). \tag{7.88}$$

In case under consideration one can introduce different definitions for such thermodynamical quantity as an entropy. Despite formal similarity definitions done may provide or not the connection of defined quantity with irreversibility of the system evolution. For example one can define the total and the partial entropy in the following way.

Definition 7.4. *The formal total entropy of nonequilibrium state is defined as*

$$S\left(\epsilon_1, \epsilon_2, t\right) = -Sp_{\{\vec{\zeta}\}} Sp_x \left\{\rho_{stc} \ln \rho_{stc}\right\}. \tag{7.89}$$

Definition 7.5. *The formal partial nonequilibrium entropy is defined as*

$$S^{(m)}\left(\epsilon_1, \epsilon_2, t\right) = -Sp_{\{\vec{\zeta}\}} Sp_x \left\{\rho_{stc}^{(m)} \ln \rho_{stc}^{(m)}\right\}. \tag{7.90}$$

It is not difficult to show that the formal partial entropy does not depend on time and has no relation to thermodynamical irreversibility.

Proposition 7.1. *For any m the formal partial entropy $S^{(m)}\left(\epsilon_1, \epsilon_2, t\right)$ is equal to zero.*

Proof. Let's consider the N-dimensional square matrix $\hat{A}$ with elements $\hat{A}_{ik} = a_i a_k$, where a_i, $i = 1, \ldots, N$ are the elements of N-dimensional vector. It is possible to find all eigenvalues λ_i and to find out the structure of eigen-subspaces for matrix A. Namely, one can show that $\lambda_1 = a_1^2 + a_2^2 + \ldots + a_N^2$, $\lambda_2 = \lambda_3 = \ldots = \lambda_N = 0$. At that eigenvector $\mathbf{e_1}$ coincides with $\mathbf{a}$, and eigen-subspace corresponding to zero eigenvalues is orthogonal to $\mathbf{a}$.

Generalizing this result on the case of infinitely dimensional matrix $\rho_{stc}^{(m)}$, one obtains: there is one eigenvector $(\pi/\Omega_{in})^{1/4} \Psi_{stc}^{(m)}\left(x, t|\{\vec{\zeta}\}\right)$, corresponding to nonzero eigenvalue $\lambda_1 = Sp_{\{\vec{\zeta}\}} Sp_x \left\{\rho_{stc}^{(m)}\right\}$, and there is an infinitely dimensional eigen-subspace, corresponding to zero eigenvalue, which is orthogonal to this vector. Supplementing the vector $(\pi/\Omega_{in})^{1/4} \Psi_{stc}^{(m)}\left(x, t|\{\vec{\zeta}\}\right)$ with any orthonormal set of vectors lying in the subspace mentioned above, one obtains the basis of the whole space which brings matrix $\rho_{stc}^{(m)}$ to diagonal form. Understanding the uncertainty $0 \ln 0$ as a limit

$$0 \ln 0 = \lim_{s \to 0} s \ln s = 0,$$

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one obtains for formal partial entropy:

$$S_f^{(m)}(\epsilon_1, \epsilon_2, t) = Sp_{\{\vec{\zeta}\}} Sp_x \left\{ \rho_{stc}^{(m)} \right\} \cdot Sp_{\{\vec{\zeta}\}} Sp_x \left\{ \ln \rho_{stc}^{(m)} \right\} = 0,$$

which makes the proof complete. $\triangle$

If one wishes to have the quantity describing irreversible behavior of the system, it is necessary to change definition of entropy.

Definition 7.6. *Total and partial entropies of nonequilibrium state are defined as*

$$S(\epsilon_1, \epsilon_2, t) = -Sp_x \left\{ \rho_{av} \ln \rho_{av} \right\}. \tag{7.91}$$

and

$$S^{(m)}(\epsilon_1, \epsilon_2, t) = -Sp_x \left\{ \rho_{av}^{(m)} \ln \rho_{av}^{(m)} \right\} \tag{7.92}$$

correspondingly, where

$$\rho_{av} = Sp_{\{\vec{\zeta}\}} \left\{ \rho_{stc} \right\}, \quad \rho_{av}^{(m)} = Sp_{\{\vec{\zeta}\}} \left\{ \rho_{stc}^{(m)} \right\}.$$

Unfortunately we have no at the moment simple enough analitical representation for the quantities defined in such a way.

To illustrate the definitions given above we calculate the average energy of oscillator in the ground, vacuum, state (i.e. at $m = 0$) assuming that both regular and stochastic parts of the external force are absent. In this case the density matrix has the form

$$\rho_{stc}^{(0)}\left(x, t; \{\vec{\zeta}\} \mid x', t'; \{\vec{\zeta}'\}\right) = \exp\left\{ -\frac{\Omega_{in}}{2}(x^2 + x'^2) - \right.$$

$$\left. -\frac{1}{2}\int_{t_1}^{t} \Phi(\tau)\, d\tau - \frac{1}{2}\int_{t_1}^{t'} \Phi^*(\tau)\, d\tau - \frac{i}{2}\left[\Phi(t) x^2 - \Phi^*(t') x'^2\right] \right\}. \tag{7.93}$$

Proposition 7.2. *Let* $\Omega(t) \equiv \Omega_{in}$, $F_0(t) \equiv 0$, $\epsilon_2 = 0$, $p_1(t) \equiv 1$. *Then the average energy*

$$E_{osc}^{(0)}(\lambda) = Sp_x Sp_{\{\vec{\zeta}\}} \left(\hat{H} \tilde{\rho}_{stc}^{(0)} \right)$$

is represented by

$$E_{osc}^{(0)}(\lambda) = \frac{1}{2}\Omega_{in}\left\{ 1 - \frac{1}{\sqrt{\lambda}}\int_{-\infty}^{+\infty} d\bar{u}_3 K_1(\lambda, \bar{u}_3) + \frac{i}{\sqrt{\lambda}}\int_{-\infty}^{+\infty} d\bar{u}_3 K_2(\lambda, \bar{u}_3) \right\}, \tag{7.94}$$

with the designations

$$K_1(\lambda, \bar{u}_3) = C_0 \bar{u}_3 \bar{q}^{st}(\lambda, \bar{u}_3) \times$$

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$$\times \left\{ -\sqrt{\frac{A_{00} - 1}{2A_{00}^2}} + \frac{\bar{u}_3}{\sqrt{\lambda} A_{00}^2} \left[\sqrt{\frac{A_{00} + 1}{2A_{00}^2}} + \frac{\bar{u}_3}{\sqrt{\lambda}} \sqrt{\frac{A_{00} - 1}{2A_{00}^2}} \right] \right\},$$

$$K_2 \left(\lambda, \bar{u}_3 \right) = C_0 \bar{u}_3 \bar{q}^{st} \left(\lambda, \bar{u}_3 \right) \times$$

$$\left\{ -\sqrt{\frac{A_{00} - 1}{2A_{00}^2}} + \frac{\bar{u}_3}{\sqrt{\lambda} A_{00}^2} \left[-\sqrt{\frac{A_{00} - 1}{2A_{00}^2}} + \frac{\bar{u}_3}{\sqrt{\lambda}} \sqrt{\frac{A_{00} - 1}{2A_{00}^2}} \right] \right\},$$

$$A_{00} \left(\lambda, \bar{u}_3 \right) = \sqrt{1 + \frac{\bar{u}_3^2}{\lambda}}. \tag{7.95}$$

The function $q^{st}(u_3)$ is an arbitrary solution of the equation

$$\epsilon_1 \frac{d^2 q^{st}}{du_3^2} + (u_3^2 + \Omega_{in}^2) \frac{dq^{st}}{du_3} + u_3 q^{st} = 0, \tag{7.96}$$

decreasing as $|u_3| \to \infty$, and the constant C_0 given by

$$C_0 = \frac{Y^{st}(u_{30}, u_{40})}{\int du_3 \; Y^{st}(u_3, 0) q^{st}(u_3)}. \tag{7.97}$$

Here the function $Y^{st}(u_3, u_4)$ is an arbitrary solution of the equation

$$\epsilon_1 \frac{d^2 Y^{st}}{du_3^2} - (u_3^2 - u_4^2 + \Omega_{in}^2) \frac{dY^{st}}{du_3} - 2u_3 u_4 \frac{dY^{st}}{du_4} - u_3 Y^{st} = 0, \tag{7.98}$$

decreasing as $u_3^2 + u_4^2 \to \infty$.

Proof. In fact the proof copies the manipulations performed repeatedly in this paper and thus may be omitted. $\triangle$

The second term inside the figure brackets in (7.94) is a level shift which is well known from quantum electrodynamics as the Lamb shift, the third term determines the magnitude of the ground state energy broadening. Note that the lifetime at this level is proportional to the inverse of the broadening

$$\Delta t^{(0)} \sim 2 \frac{\sqrt{\lambda}}{\Omega_{in}^2} \left\{ \int\limits_{-\infty}^{+\infty} d\bar{u}_3 K_2 \left(\lambda, \bar{u}_3 \right) \right\}^{-1}. \tag{7.99}$$

The average energy of a randomly wandering (QHO) for any quantum level is calculated quite similarly.

Conclusion

There are three different reasons which may cause a chaos in the basic quantum mechanical object, i.e. the wave function. The first reason refers to measurements performed over a quantum system [4, 5]. The second reason consists in the more fundamental openness of any quantum system resulting from the fact that all the beings are immersed into a physical vacuum [14]. In the third place, as it follows from the recent papers [6, 7, 8], a chaos may also appear in the wave function even in a closed dynamical system. As it is shown in [5], there is a close connection between a classical nonintegrability and a chaos in the corresponding quantum system. Many of the fundamentally important questions of the quantum physics such as the Lamb shift of energy levels, spontaneous transitions between the atom levels, quantum Zeno effect [17], processes of chaos and self-organization in quantum systems, especially those where the phenomena of phase transitions type may occur, can be described qualitatively and quantitatively in a rigorous way only within the nonperturbative approaches. The Lindblad representation [15, 16] for the density matrix of the system "quantum object + thermostat" describes *a priori* the most general situation which may appear in the nonrelativistic quantum mechanics. Nevertheless, we need to consider a reduced density matrix on a semi-group [4], when investigating a quantum subsystem. This is quite an ambiguous procedure and moreover its technical realization is possible only in the framework of a particular perturbative scheme.

A crucially new approach to constructing the quantum mechanics of the closed nonrelativistic system "quantum object + thermostat" has been developed recently by the authors of [6, 7] from the principle of "local correctness of Schrödinger representation". To put it differently, it has been assumed that the evolution of the quantum system is such that it may be described by the Schrödinger equation on any small time interval, while the motion as a whole is described by a SDE for the wave function. In this case, however, there emerges not a simple problem to find a measure for calculating the average values of the physical system parameters. Nevertheless, there exists a certain class of models for which all the derivations can be made not applying the perturbation theory [7].

In the present paper we explore further the possibility of building the nonrelativistic quantum mechanics of closed system "quantum object + thermostat" within the framework of the model of one-dimensional randomly wandering QHO (with a random frequency and subjected to a random external force). Mathematically the problem is formulated in terms of SDE for a complex-valued probability process defined on the extended space $R^1 \otimes R_{\{\xi\}}$. The initial SDE is reduced to the Schrödinger

equation for an autonomous oscillator defined on a random space-time continuum, with the use of a nonlinear transformation and one-dimensional etalon nonlinear equation of the Langevin type defined on the functional space $R_{\{\xi\}}$. It is possible to find for any fixed $\{\xi\}$ an orthonormal basis of complex-valued random functionals in the space $L_2(R^1)$ of square-integrable functions. With the assumption that the random force generator is described by a white noise correlator, the Fokker-Planck equation for a conditional probability is found. From the solutions of this equation on an infinitely small time interval a measure of the functional space $R_{\{\xi\}}$ can be constructed. Then by averaging an instantaneous value of the transition probability over the space $R_{\{\xi\}}$, the mean value of the transition probability is represented by a functional integral. Using the generalized Feynman-Kac theorem, it is possible to reduce the functional integral in the most general case, where both frequency and force are random, to a multiple integral of the fundamental solution of some parabolic partial differential equation. The qualitative analysis of the parabolic equation shows that it may have discontinuous solutions [18]. This is equivalent to the existence of phenomena like the phase transitions in the microscopic transition probabilities. In the context of the developed approach the representation of the stochastic density matrix is introduced, which allows to build a closed scheme for both nonequilibrium and equilibrium thermodynamics. The analytic formulas for the ground energy level broadening and shift are obtained, as well as for the entropy of the ground quantum state.

The further development of the considered formalism in application to exactly solvable many-dimensional models may essentially extend our understanding of the quantum world and lead us to the new nontrivial discoveries.

Appendix

Theorem A.1. *Let the set of random processes* $(\xi_1, \xi_2, .., \xi_n) \equiv \vec{\xi}$ *satisfy the set of SDE*

$$\dot{\xi}_i(t) = a_i\left(\vec{\xi}, t\right) + \sum_{j=1}^{n} b_{ij}\left(\vec{\xi}, t\right) f_j(t), \quad i = 1, ..., n,$$

$$\langle f_i(t) f_j(t') \rangle = \delta_{ij}\delta(t - t'),$$

so that the Fokker-Planck equation for the conditional transition probability density

$$P^{(2)}\left(\vec{\xi}_2, t_2 | \vec{\xi}_1, t_1\right) = \left\langle \delta\left(\vec{\xi}(t_2) - \vec{\xi}_2\right)\right\rangle\Big|_{\vec{\xi}(t_1)=\vec{\xi}_1} \quad t_2 > t_1 \qquad \text{(A.1)}$$

is given by

$$\frac{\partial P^{(2)}}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} \left(a_i P^{(2)} \right) + \sum_{i,j,l,n} \frac{\partial}{\partial \xi_i} \left(b_{li} \frac{\partial}{\partial \xi_n} \left(b_{nj} P^{(2)} \right) \right) \equiv \hat{L}^{(n)}(\vec{\xi}) P^{(2)}. \qquad (A.2)$$

The processes ξ_i are assumed to be markovian and satisfy the condition $\vec{\xi}(t_0) = \vec{\xi}_0$.
At the same time the function (A.1) gives their exhaustive description:

$$P^{(n)}\left(\vec{\xi}_n, t_n; \dots \vec{\xi}_1, t_1; \vec{\xi}_0, t_0 \right) = P^{(2)}\left(\vec{\xi}_n, t_n | \vec{\xi}_{n-1}, t_{n-1} \right) \dots P^{(2)}\left(\vec{\xi}_1, t_1 | \vec{\xi}_0, t_0 \right) \qquad (A.3)$$

where $P^{(n)}$ is a density of the probability that the trajectory $\vec{\xi}(t)$ would pass through
the sequence of intervals $\left[\vec{\xi}_1, \vec{\xi}_1 + d\vec{\xi}_1 \right], \dots, \left[\vec{\xi}_n, \vec{\xi}_n + d\vec{\xi}_n \right]$ at the subsequent moments
of time $t_1 < \dots < t_n$, respectively.

    Under these assumptions we can obtain the following representation for an av-
eraging procedure

$$\left\langle \exp\left\{ -\int_{t_0}^{t} V_1\left(\vec{\xi}(\tau), \vec{\xi}(t) \right) d\tau - V_2\left(\vec{\xi}(t) \right) \right\} \right\rangle = \int d\vec{\xi} e^{-V_2(\vec{\xi},t)} Q\left(\vec{\xi}, \vec{\xi}, t \right), \qquad (A.4)$$

where $d\vec{\xi} = d\xi_1 \dots d\xi_n$, and the function $Q\left(\vec{\xi}, \vec{\xi}', t \right)$ is a solution of the problem

$$\frac{\partial Q}{\partial t} = \left[\hat{L}^{(n)}(\vec{\xi}) - V_1(\vec{\xi}, \vec{\xi}') \right] Q, \qquad (A.5)$$

$$Q\left(\vec{\xi}, \vec{\xi}', t \right) \xrightarrow[t \to t_0]{} \delta\left(\vec{\xi} - \vec{\xi}_0 \right), \quad Q\left(\vec{\xi}, \vec{\xi}', t \right) \xrightarrow[\|\vec{\xi}\| \to \infty]{} 0,$$

where $\| \cdot \|$ is a norm in R^n.
Proof. The proof is performed formally under the assumption that all the manip-
ulations are legal. Denote the left side of the equality (A.4) by I and expand the
averaging quantity into the Tailor series:

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n(t), \qquad (A.6)$$

where

$$\mu_n(t) = \left\langle \left\{ \int_{t_0}^{t} V_1(\tau) d\tau + V_2(t) \right\}^n \right\rangle =$$

$$= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \left\langle V_2^{n-m}(t) \left\{ \int_{t_0}^{t} V_1(\tau) d\tau \right\}^m \right\rangle = \qquad (A.7)$$

$$= \sum_{m=0}^{n} \frac{n!}{(n-m)!} \left\langle V_2^{n-m}(t) \int_{t_0}^{t} d\tau_m \int_{t_0}^{\tau_m} d\tau_{m-1} \ldots \int_{t_0}^{\tau_2} d\tau_1 V_1(\tau_m) \ldots V_1(\tau_1) \right\rangle$$

For brevity sake in (A.7) there was introduced the designation $V_1(\tau) \equiv V_1\left(\vec{\xi}(\tau), \vec{\xi}(t)\right)$, $V_2(t) \equiv V_2\left(\vec{\xi}(t)\right)$. Using the Fubini theorem, we can represent the averaging procedure in (2.18) as an integration with the weight $P^{(n)}$ from (A.3)

$$\left\langle V_2^{n-m}(t) \int_{t_0}^{t} d\tau_m \int_{t_0}^{\tau_m} d\tau_{m-1} \ldots \int_{t_0}^{\tau_2} d\tau_1 V_1(\tau_m) \ldots V_1(\tau_1) \right\rangle =$$

$$= \int d\vec{\xi} \int d\vec{\xi}_m \ldots \int d\vec{\xi}_1 \int_{t_0}^{t} d\tau_m \ldots \int_{t_0}^{\tau_2} d\tau_1 P^{(2)}\left(\vec{\xi}, t \big| \vec{\xi}_m, t_m\right) P^{(2)}\left(\vec{\xi}_m, t_m \big| \vec{\xi}_{m-1}, t_{m-1}\right) \ldots$$

$$\ldots P^{(2)}\left(\vec{\xi}_2, t_2 \big| \vec{\xi}_1, t_1\right) V_2^{n-m}\left(\vec{\xi}\right) V_1\left(\vec{\xi}_m, \vec{\xi}\right) \ldots V_1\left(\vec{\xi}_1, \vec{\xi}\right).$$

Changing, where it is necessary, the order of integration we can obtain the following representation for the n-th moment $\mu_n(t)$:

$$\mu_n(t) = \sum_{m=0}^{n} \frac{n!}{(n-m)!} \int d\vec{\xi} V_2^{n-m}\left(\vec{\xi}\right) Q_m\left(\vec{\xi}, \vec{\xi}', t\right), \qquad (A.8)$$

where the countable set of functions $Q_m\left(\vec{\xi}, \vec{\xi}', t\right)$ is determined from the recurrence relations

$$Q_m\left(\vec{\xi}, \vec{\xi}', t\right) = \int_{t_0}^{t} d\tau \int d\vec{\eta} \, V_2^{n-m}\left(\vec{\xi}\right) P^{(2)}\left(\vec{\xi}', t \big| \vec{\eta}, \tau\right) V_1\left(\vec{\eta}, \vec{\xi}'\right) Q_{m-1}\left(\vec{\eta}, \vec{\xi}', \tau\right), \quad (A.9)$$

$$m = 0, 1, 2, \ldots,$$

where

$$Q_0\left(\vec{\xi}, \vec{\xi}', t\right) = P^{(2)}\left(\vec{\xi}, t \big| \vec{\xi}_0, t_0\right) \qquad (A.10)$$

i.e. in fact the function Q_0 is independent of $\vec{\xi}'$. Upon the substitution of (A.8) into (A.6) we insert the summation procedure under the integration sign and then, changing the order of double summation, get the expression

$$I = \int d\vec{\xi} \, e^{-V_2\left(\vec{\xi}, t\right)} Q\left(\vec{\xi}, \vec{\xi}, t\right), \qquad (A.11)$$

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where
$$Q\left(\vec{\xi},\vec{\xi'},t\right) = \sum_{n=0}^{\infty}(-1)^n Q_n\left(\vec{\xi},\vec{\xi'},t\right). \tag{A.12}$$

The representation (A.4) is thus obtained.

It remains to prove that the function Q from (A.11) is a solution of the problem (A.5). Using (A.12) and (A.9) we can easily show that Q satisfies the integral equation

$$Q\left(\vec{\xi},\vec{\xi'},t\right) + \int_{t_0}^{t} d\tau \int d\vec{\eta}\, P^{(2)}\left(\vec{\xi},t\middle|\vec{\eta},\tau\right) V_1\left(\vec{\eta},\vec{\xi'}\right) Q\left(\vec{\eta},\vec{\xi'},\tau\right) = Q_0\left(\vec{\xi},t\right) \tag{A.13}$$

Taking into account that Q_0 satisfies (A.2) and the initial condition $Q_0\left(\vec{\xi},t_0\right) = \delta\left(\vec{\xi}-\vec{\xi_0}\right)$ and is an integrable function, it can be deduced from (A.13) that Q coincides with the solution of the problem (A.5). The representation (A.4), (A.5) is thus obtained. $\triangle$

References

[1] Proceedings of Adriatico Research Conference and Miniworkshop *Quantum Chaos*, 4June–6July 1990, Trieste, Italy

[2] C.Presilla, R.Onofrio, U.Tambini, Ann.Phys., v. 248, p. 95 (1996)

[3] C.W.Gardiner, M.J.Collett, Phys.Rev. A, v. 31, p. 3761 (1985)

[4] N.Gisin, I.C.Percival, J.Phys. A, v. 25, p. 5677 (1992)

[5] N.Knauf, Y.G.Sinai, e-print N 232 http://www.math.tu-berlin.de

[6] A.V.Bogdanov, A.S.Gevorkyan, Proceedings of Int. Workshop on Quantum Systems, Minsk, Belarus, p. 26 (1996)

[7] A.V.Bogdanov, A.S.Gevorkyan, A.G.Grigoryan, AMS/IP Studies in Advanced Mathematics, v. 13, p. 81 (1999)

[8] A.V.Bogdanov, A.S.Gevorkyan, A.G.Grigoryan, S.A.Matveev, Int. Journ. Bifurcation and Chaos, v. 9, N. 12, p. 9 (1999)

[9] A. N. Baz', Ya. B. Zel'dovich and A. M. Perelomov, *Scattering reactions and Decays in Nonrelativistic Quantum Mechanics,* (in Russia), "Nauka", Moscow, 1971.

[10] I.M.Lifshitz, S.A.Gredeskul and L.P.Pastur, *Introduction to the Theory of Non-Regular Systems,* (in Russia), "Nauka", Moscow, 1982.

[11] C.W.Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences,* Springer-Verlag Berlin New-York Tokyo, 1985

[12] Kac M. *Probability and Related Topics in Phisical Sciences,* New York: Interscience Publishers, 1959.

[13] D.N.Zubarev, *Nonequilibrium statistical thermodinamics,* Nauka, 1971 (in russian).

[14] J.Glimm, A.Jaffe, *Quantum Physics. A Functional Integral Point of View,* Springer-Verlag, 1981.

[15] V.Gorini, A.Kossakowski and E.C.G.Sudarshan, *J.Math.Phys.,* v. 17, p. 821 (1976).

[16] G.Lindblad, *Comm.Math.phys.,* v. 48, p. 119 (1976).

[17] W.M.Itano, D.J.Heinzen, J.J.Bollinger and D.J.Wineland, *Phys.Rev. A,* v. 41, p. 2295 (1990).

[18] D.Joseph, *Stability of fluid motions,* Springer-Verlag, (1976).